THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2016–2017)
Introduction to Topology
Exercise 10 Connectedness

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Show that an infinite set X with the cofinite topology is connected. Why is the word "infinite" is needed?

2. Given two topologies $\mathcal{T}_1 \subset \mathcal{T}_2$ for a nonempty set X. What is the implication of connectedness about $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$?

3. Let $(X, \mathcal{T})$ be connected and $\emptyset \neq A \subsetneq X$.

   (a) It is known that if A is connected then $\overline{A}$ is so. Give a counter-example of the converse.

   (b) If A is connected, is it necessary true that $\operatorname{Int}(A)$ is so?

   (c) Prove that $\operatorname{Frt}(A) \neq \emptyset$. That is, there exists $x \in X$ such that every neighborhood of x intersects A and $X \setminus A$.

   Is $\operatorname{Frt}(A)$ always connected if A is connected? What about the connectedness of A if $\operatorname{Frt}(A)$ is connected?

4. Let $A \subset X$. Show that any connected set intersecting both A and $X \setminus A$ must also intersect their common frontier, $\operatorname{Frt}(A)$.

5. Prove that the definitions of a connected component are well-defined and are equivalent to each other.

6. Let X be a space that every connected component is a singleton. Give two examples of such a space.

7. Let $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous function and X is connected. Prove that its graph $G_f = \{ (x, f(x)) \in X \times Y : x \in X \}$ is a connected subset of the product space $X \times Y$. Do you think the converse is true?

8. Let $f, g\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous functions and X is connected. Show that if there exists $x_0 \in X$ such that $f(x_0) = g(x_0)$ then $G_f \cup G_g$ is connected. Is the converse true?

9. Let X, Y be connected spaces and $f\colon (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$, $g\colon (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$ be continuous. Construct a quotient space $(X \sqcup Y)/\sim$ by $x \sim y$ if $f(x) = g(y)$. Show that if f or g is surjective, then $(X \sqcup Y)/\sim$ is connected.

   Remark. The result is intuitively obvious but finding a clean proof is the essence.

10. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $L_\alpha = \{\, \mathbf{x} \in \mathbb{R}^n \;:\; f(\mathbf{x}) = \alpha \,\}$, i.e., the level set wrt α.

    (a) If $A = L_\alpha \cup L_\beta$ with $\alpha \neq \beta$, show that A is disconnected.

    (b) Is it true that L_α is always connected?

11. Prove the two variations of connectedness theorem:

    (a) Let A_α be a family of connected subsets in X and there is a connected subset C such that $C \cap A_\alpha \neq \emptyset$ for each α. Then $C \cup (\bigcup_\alpha A_\alpha)$ is also connected.

    (b) Let A_n be a countable family of connected subsets in X such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Then $\bigcup_n A_n$ is also connected.

12. Let X, Y be connected spaces and $A \subsetneq X$, $B \subsetneq Y$. Prove that $(X \times Y) \setminus (A \times B)$ is connected.

13. Let $f\colon X \to Y$ be a mapping such that Y is having the quotient topology induced by f and is connected. Prove that if for all $y \in Y$, the subset $f^{-1}(y) \subset X$ is connected, then X is connected. Apply this result to show that $U(n)$, the unitary group, is connected.